

Optimal nonlinear filtering of quantum state

V.I. Man'ko*

*P.N. Lebedev Physical Institute,
Russian Academy of Sciences, Leninskii Prospect 53,
Moscow 119991, Russia
Moscow Institute of Physics and Technology
Institutskii Per. 9, Dolgoprudny Moscow Region 141700, Russia*

L.A. Markovich†

*Moscow Institute of Physics and Technology
Institutskii Per. 9, Dolgoprudny Moscow Region 141700, Russia
V.A. Trapeznikov Institute of Control Sciences, Moscow,
Profsoyuznaya 65, 117997 Moscow, Russia
Institute for information transmission problems, Moscow,
Bolshoy Karetny per. 19,
build.1, Moscow 127051, Russia*

(Dated: January 24, 2017)

We extend the optimal filtering equation known from the Stratonovich filtering theory on the quantum process case. The used observation model is based on an indirect measurement method, where the measurement is performed on an ancilla system that is interacted with an unknown one. Observation model for single qudit system is proposed.

PACS numbers: 03.65.Ud, 03.67.Mn

I. INTRODUCTION

There has been recent interest in quantum filtering and state estimation problems in quantum information theory and quantum control [1–3]. Last experimental and theoretical advances in quantum technology provide a strong motivation for the quantum control including the engineering of quantum states, the stability theory, the quantum error correction, the robust control and quantum networks [4, 13–16]. Also, the quantum control plays a fundamental role in the development of new quantum technologies like the quantum computation. For example, the quantum filtering in coherent states is introduced in [17] and the tomographical approach is used for the signal recognition and denoising in used in a number of recent papers [18–20].

It is known, that for the quantum state estimation one has to give the measurement strategy that is used to get information, and the estimator mapping the measurement data to the state space. In this paper, we consider the weak measurement model where an unobservable quantum system is coupled with a ancilla system that can be measured (a probe system). A von Neumann measurement is used, described by a set of projection operators $\{P_n = |n\rangle\langle n|\}$. Every operator describes what happens on one of possible outcomes of the measurement. The method is based on the set of copies of

the same system being in the same state. It is obvious that in practice many identical copies of the system are impossible to implement and hence, this method can not be applied to the dynamically evolving states, as in the quantum control [26]. The measurement scheme described above is based on two-qubit systems. This is due to fast development of quantum technologies such as the quantum computers, quantum cryptography and teleportation. These technologies promise in the future to lead to a revolution in the technology and the communication.

The quantum computing is built from elementary processing elements, namely, quantum bits - qubits. It is known that classical computers elements (bits) take only two values, logic zero and logic one. On the contrary, qubits as quantum objects can be located also in a coherent superposition of these two states. Thus they describe the intermediate state between the logic zero and one. By measuring the qubit, we get zero or one with some probabilities. The quantum computers will be able to a finite time to solve problems, to the solution of which the classical supercomputers will take an unacceptable time. "Breaking" of a cryptographic RSA algorithm based on finding the decomposition of large numbers into primes is the famous example. The classical computer solving a similar problem using brute force, would have spent an enormous time comparable to the lifetime of the universe, while quantum computing system can solve it within several minutes. An implementation of such computer has several obstacles. Quantum states of ions, electrons and Josephson junctions used as qubits are extremely unstable and can not be stored in one state for a long time. For the implementation of computational algorithms one

* manko@lebedev.ru

† kimo1@mail.ru

needs a set of interacted qubits in a particular state. In recent years, the stability of the states of qubits was increased from nanoseconds to milliseconds, but this task is still extremely complicated, especially for the multipartite systems with a large number of qubits.

One way to resolve this problem is to use a single qudit system as the quantum objects instead of qubits. Such single qudit systems have more states than two-qubit ones. Thus, the dimension of the system is greatly reduced. A wide number of papers exists devoted to the study of various kinds of characteristics of quantum correlations in systems with subsystems such as the two qubit system. This quantum system can have a correlation responsible for the entanglement phenomenon [27] and for the violation of Bell's inequality [28]. These correlations may also be responsible for the quantum discord [29, 30]. However, in literature the systems without subsystems (one qudit, qudit) are paid less attention. Recently, in [31–34] it was shown that the quantum properties of systems without subsystems can be formulated using the method of an invertible mapping. In [34] the entanglement concept and correlations in the single qudit state are discussed. Using the latter mapping, the notion of the separability and the entanglement was extended in [35] to the case of the single qudit X -state with $j = 3/2$.

II. CONTRIBUTIONS OF THIS PAPER

The problems related to the quantum state estimation and the quantum state filtering are fundamental for the quantum information theory and quantum control. In [42] the well-known procedure in the field of classical control theory, namely the Kalman filtration [39] was applied to the quantum filtering area. The problem of the filtering of unknown signals from the mixture with noise is well studied in classical probability theory. The Kalman filter is known provides the optimal solution for the linear recursive model of the observation. However, for non-linear models that appear in practice the Kalman filter is not applicable.

The aim of this work is to propose the filtering method optimal for non-linear quantum processes. To this end, the general filtering equation introduced in [43, 44] was extended to the quantum observation model for the two-qubit system. In author's paper [45] it is proved that the optimal filtering equation is nothing else but the Kalman filter in the case of the linear model. Note, that the optimal filtering equation does not contain the explicit probabilistic characteristics of the unknown unobservable sequence. This allows us to find the optimal state estimate knowing only observable quantities.

A further aim is to propose a quantum measurement model for the single qudit state. The optimal filtering method proposed for the two-qubit models can be extended to the latter case. The construction of such type of the observation models is useful in the light of possible practical use of the single qudit systems.

The paper is organized as follows. In Sec. III we give the brief overview on the notion of the weak measurement for the system of two qubits. In Sec. IV the method of the invertible mapping is used to obtain the non-linear measurement model for the single qudit system. The physical meaning of the correlations in such system is given. In the next section the optimal filtering equation is described. The latter approach is extended to the case of the nonlinear quantum models. It is shown that by means of the optimal filtering equation one can find the optimal solution of the nonlinear quantum filtering problem without using any linearization procedures like in [42]. In Sec. VI the observation model is rewritten in term of the tomogram. The latter approach is used to write the Shannon entropies and information depending on a time step of the observation model. Thus, it is possible to observe the time evolution of the information of the quantum nonlinear process. The Werner state example of the information time evolution is provided.

III. THE WEAK MEASUREMENT

In spirit of [42] let us consider the discrete time case of the indirect measurement. We suppose that the unobservable and the observable measurements of quantum systems are quantum bits. Under the weak measurement we mean that the projective measurements are done on the extra ancilla system that is in state $\theta_M = (\theta_{M_1}, \theta_{M_2}, \theta_{M_3})$ coupled with the system $\theta_S = (\theta_{S_1}, \theta_{S_2}, \theta_{S_3})$ that we are interested in. Two Bloch vector representations of the latter states are

$$\begin{aligned}\rho_M(k) &= (I + \theta_M(k)\sigma^M)/2, \\ \rho_S(k) &= (I + \theta_S(k)\sigma^S)/2,\end{aligned}\tag{1}$$

where the σ^S and σ^M are symbolic vectors constructed from the Pauli operators acting on Hilbert spaces H_S and H_M , respectively. The indirect measurement is proceeded by the following way. At the time step k we prepare the ancilla qubit in a known state. We couple it to an unknown system. The composite system is represented by the 4-dimensional square density matrix $\rho_{S+M}(k)$. Let us take it as a direct product of two later states, i.e. $\rho_{S+M}(k) = \rho_S(k) \otimes \rho_M(k)$. Both qubits evolve according to bipartite dynamics at sampling time h . At the end we do the von Neumann measurement on the ancilla qubit. Generally, the von Neumann measurement is the measurement of Pauli operators. For example, if we are interested in the measurement of the observable σ_x , then possible outcomes are its eigenvalues (± 1). The algorithm is repeated at the next time step $k + 1$.

The probabilities of two different outcomes

$$\begin{aligned}A_x &= I \times \sigma_x = \left(\frac{1}{2}\right) |1\rangle\langle 1| + \left(-\frac{1}{2}\right) |2\rangle\langle 2|, \\ A_x |1\rangle\langle 1| &= \left(\frac{1}{2}\right) |1\rangle\langle 1|, \quad A_x |2\rangle\langle 2| = \left(-\frac{1}{2}\right) |2\rangle\langle 2|\end{aligned}$$

of the von Neumann measurement are the following

$$P(+1) = \left(\frac{1}{2}\right) |1\rangle\langle 1| = I \times \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{Tr} P = 2,$$

$$P(-1) = \left(-\frac{1}{2}\right) |2\rangle\langle 2| = I \times \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad P^2 = P.$$

The evolution of the system is controlled by the unitary operator. Let the matrix of this operator has the basic view

$$W = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix}. \quad (2)$$

The state of the composite system after the interaction is the following

$$\rho_{S+M}(k+1) = W \rho_{S+M}(k) W^\dagger$$

and the reduced density matrix of the system we are interested in is

$$\begin{aligned} \rho_S(k+1) &= \text{Tr}_M \rho_{S+M}(k+1) \\ &= \text{Tr}_M W \rho_S(k) \otimes \rho_M(k) W^\dagger. \end{aligned}$$

Since, we are interested in measuring σ_x , the states after the measurement are

$$\rho(\pm 1) = \frac{\rho_{S+M} P_\pm}{\text{Tr}(\rho_{S+M} P_\pm)}$$

and the eigenstates of the measurement are

$$\begin{aligned} \theta_S(\pm 1) &= \frac{\text{Tr}_M(\rho_{S+M} P_\pm)}{\text{Tr}(\text{Tr}_M(\rho_{S+M} P_\pm))} \\ &= \frac{\tilde{\rho}_{S\pm}}{\text{Tr}(\text{Tr}_M(\rho_{S+M} P_\pm))}. \end{aligned} \quad (3)$$

A. Examples of the evolution matrices

As an example, the evolution matrix can be taken as $W = e^{-ih(a_y \sigma_y^S \otimes \sigma_y^M)}$ (see [42]), where a_y is the coupling parameter and h is the sampling time. For the measurement $A_x = I \otimes \sigma_x$ and $a_y h = \pi/2$, the probabilities of two different outcomes are

$$P(+1) = (1 + \theta_{S_2} \theta_{M_3}), \quad P(-1) = (1 - \theta_{S_2} \theta_{M_3}).$$

The post measurement states are the following

$$\theta_S(\pm 1) = \begin{pmatrix} \frac{\theta_{S_3} \theta_{M_2} \pm \theta_{S_1} \theta_{M_1}}{1 \pm \theta_{S_2} \theta_{M_3}} \\ \frac{\theta_{S_2} \pm \theta_{M_3}}{1 \pm \theta_{S_2} \theta_{M_3}} \\ \frac{\pm \theta_{S_3} \theta_{M_1} - \theta_{S_1} \theta_{M_2}}{1 \pm \theta_{S_1} \theta_{M_2}} \end{pmatrix}. \quad (4)$$

Since the probability of the new state depends on both measurements θ_S and θ_M we can retrieve the information about the useful state using only the observable measurement.

Let us select the unitary evolution matrix of the following view

$$W = \begin{pmatrix} \cos \varphi & 0 & 0 & \sin \varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin \varphi & 0 & 0 & \cos \varphi \end{pmatrix} \quad (5)$$

The numerator and the denominator elements of (3) are given in appendix. For simplicity, the angle φ can be selected equal to zero and then the elements of (3) are

$$\begin{aligned} \widetilde{\rho}_S(1, 1) &= (\theta_{m_1} + 1)(\theta_{s_3} + 1)/4, \\ \widetilde{\rho}_S(1, 2) &= (\theta_{s_1} - \theta_{s_2} i)(\theta_{m_1} + 1)/4, \\ \widetilde{\rho}_S(2, 1) &= (\theta_{s_1} + \theta_{s_2} i)(\theta_{m_1} + 1)/4, \\ \widetilde{\rho}_S(2, 2) &= -(\theta_{m_1} + 1)(\theta_{s_3} - 1)/4 \end{aligned}$$

and the denominator is $(\theta_{m_1} + 1)/2$.

For the special case of the evolution matrix that depends only on one angle we get

$$W = \begin{pmatrix} \cos \varphi & 0 & 0 & \sin \varphi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sin \varphi & 0 & 0 & \cos \varphi \end{pmatrix}. \quad (6)$$

Elements of (3) are given in appendix. We can select, for example, the angle $\varphi = 0$ and write

$$\begin{aligned} \widetilde{\rho}_S(1, 1) &= (\theta_{m_3} + 1)(\theta_{s_3} + 1)/4, \\ \widetilde{\rho}_S(1, 2) &= (\theta_{m_1} - \theta_{m_2} i)(\theta_{s_1} - \theta_{s_2} i)/4, \\ \widetilde{\rho}_S(2, 1) &= (\theta_{m_1} + \theta_{m_2} i)(\theta_{s_1} + \theta_{s_2} i)/4, \\ \widetilde{\rho}_S(2, 2) &= (\theta_{m_3} - 1)(\theta_{s_3} - 1)/4. \end{aligned} \quad (7)$$

The denominator is then $(\theta_{m_3} \theta_{s_3} + 1)/4$.

IV. SINGLE QUDIT OBSERVATION MODEL

Recently, it was observed in [31–33] that the quantum properties of the systems without subsystems can be formulated using the invertible map of integers $1, 2, 3, \dots$ onto the pairs (triples, etc) of integers (i, k) , $i, k = 1, 2, \dots$ (or semiintegers). For example, the single qudit state $j = 0, 1/2, 1, 3/2, 2, \dots$ can be mapped onto the density operator of the system containing the subsystems like the state of two qudits.

Let the quantum state in four dimensional Hilber space \mathcal{H} be described by the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \quad (8)$$

that $\rho = \rho^\dagger$, $\text{Tr} \rho = 1$ and its eigenvalues are nonnegative.

The latter matrix can describe the two-qubit system. To this end, let us use the following invertible mapping $1 \leftrightarrow 1/2 \ 1/2$; $2 \leftrightarrow 1/2 \ -1/2$; $3 \leftrightarrow -1/2 \ 1/2$; $4 \leftrightarrow -1/2 \ -1/2$ and rewrite the density matrix ρ as

$$\rho_{1/2} = \begin{pmatrix} \rho_{1/2} \ 1/2, 1/2 \ 1/2 & \rho_{1/2} \ 1/2, 1/2 \ -1/2 & \rho_{1/2} \ 1/2, -1/2 \ 1/2 & \rho_{1/2} \ 1/2, -1/2 \ -1/2 \\ \rho_{1/2} \ -1/2, 1/2 \ 1/2 & \rho_{1/2} \ -1/2, 1/2 \ -1/2 & \rho_{1/2} \ -1/2, -1/2 \ 1/2 & \rho_{1/2} \ -1/2, -1/2 \ -1/2 \\ \rho_{-1/2} \ 1/2, 1/2 \ 1/2 & \rho_{-1/2} \ 1/2, 1/2 \ -1/2 & \rho_{-1/2} \ 1/2, -1/2 \ 1/2 & \rho_{-1/2} \ 1/2, -1/2 \ -1/2 \\ \rho_{-1/2} \ -1/2, 1/2 \ 1/2 & \rho_{-1/2} \ -1/2, 1/2 \ -1/2 & \rho_{-1/2} \ -1/2, -1/2 \ 1/2 & \rho_{-1/2} \ -1/2, -1/2 \ -1/2 \end{pmatrix}. \quad (9)$$

Let $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$, where \mathcal{H}^1 and \mathcal{H}^2 are the two dimensional Hilbert spaces. The reduced density matrices

of the twoqubit system (the subsystems) can be defined using the partial trace of (9) as

$$\rho_1 = \begin{pmatrix} \rho_{1/2} \ 1/2, 1/2 \ 1/2 + \rho_{1/2} \ -1/2, 1/2 \ -1/2 & \rho_{1/2} \ 1/2, -1/2 \ 1/2 + \rho_{1/2} \ -1/2, -1/2 \ -1/2 \\ \rho_{-1/2} \ 1/2, 1/2 \ 1/2 + \rho_{-1/2} \ -1/2, 1/2 \ -1/2 & \rho_{-1/2} \ 1/2, -1/2 \ 1/2 + \rho_{-1/2} \ -1/2, -1/2 \ -1/2 \end{pmatrix}, \quad (10)$$

$$\rho_2 = \begin{pmatrix} \rho_{1/2} \ 1/2, 1/2 \ 1/2 + \rho_{-1/2} \ 1/2, -1/2 \ 1/2 & \rho_{1/2} \ 1/2, 1/2 \ -1/2 + \rho_{-1/2} \ 1/2, -1/2 \ -1/2 \\ \rho_{1/2} \ -1/2, 1/2 \ 1/2 + \rho_{-1/2} \ -1/2, -1/2 \ 1/2 & \rho_{1/2} \ -1/2, 1/2 \ -1/2 + \rho_{-1/2} \ -1/2, -1/2 \ -1/2 \end{pmatrix}. \quad (11)$$

The density matrix $\rho_{1/2}$ can also describe the single qudit system if we rewrite it using the following invertible mapping $1 \leftrightarrow 3/2$, $2 \leftrightarrow 1/2$, $3 \leftrightarrow -1/2$, $4 \leftrightarrow -3/2$ as

$$\rho_{3/2} = \begin{pmatrix} \rho_{3/2, 3/2} & \rho_{3/2, 1/2} & \rho_{3/2, -1/2} & \rho_{3/2, -3/2} \\ \rho_{1/2, 3/2} & \rho_{1/2, 1/2} & \rho_{1/2, -1/2} & \rho_{1/2, -3/2} \\ \rho_{-1/2, 3/2} & \rho_{-1/2, 1/2} & \rho_{-1/2, -1/2} & \rho_{-1/2, -3/2} \\ \rho_{-3/2, 3/2} & \rho_{-3/2, 1/2} & \rho_{-3/2, -1/2} & \rho_{-3/2, -3/2} \end{pmatrix}.$$

The matrix saves the standard properties of the density matrix, i.e. $\rho_{3/2} = \rho_{3/2}^\dagger$, $\text{Tr} \rho_{3/2} = 1$ hold and its eigenvalues are nonnegative. Using the partial trace two "artificial subsystems" ("artificial qubits") can be introduced. It means that all equalities and inequalities known for matrix $\rho_{1/2}$ (the two-qubit system) are valid for matrix $\rho_{3/2}$ (the single qudit system).

Hence, one can think that density matrix $\rho_{S+M}(k)$ describes the single qudit state. Applying the latter method, the two "artificial subsystems" $\rho_S(k)$ and $\rho_M(k)$ can be constructed such that $\rho_{S+M}(k) = \rho_S(k) \otimes \rho_M(k)$. Then one can construct the observation model as for the two-qubit system. However, it does not contain the real observable ancilla qubit and the unobservable one.

A. Physical meaning of the "artificial qubits"

Let us start from the short example of two coins which can drop on the one (1) or on the second (2) side. Hence, there are two random variables m_1 , m_2 and four opportunities $(m_1, m_2) = \{(11), (12), (21), (22)\}$ with probabilities p_{ij} , $i, j = 1, 2$. Let $\omega(m_1, m_2)$ be the joint probability function of these two random variables. Their marginal probability functions can be defined as

$$\omega_1(m_1) = \sum_{m_2} \omega(m_1, m_2), \quad \omega_2(m_2) = \sum_{m_1} \omega(m_1, m_2).$$

Hence, the correlation between the two observations is given by

$$\langle m_1, m_2 \rangle = \sum_{m_1, m_2} m_1 m_2 \omega(m_1, m_2).$$

However, we can be interested not in the whole system, but only in cases when the first coin falls on the one side and the second coin is not interesting for us. We have two new outcomes $\{\tilde{\omega}\}$ with probabilities $\tilde{p}_1 = p_{11} + p_{12}$ and $\tilde{p}_2 = p_{22} + p_{21}$. Analogically, if we are interested only in the second coin, we have two new outcomes ω with probabilities $p_1 = p_{11} + p_{21}$ and $p_2 = p_{22} + p_{12}$. Outcomes $\{\omega\}$ and $\{\tilde{\omega}\}$ are correlated. The latter example is helpful to show the existence of correlations in systems without subsystems.

If we have a single qudit system with the spin $j = 3/2$, we can write the sample space Ω of four outcomes $\omega \in \Omega$, the values of the spin projections $|m\rangle = \{|3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle\}$ with probabilities $p_{3/2}, p_{1/2}, p_{-1/2}, p_{-3/2}$. Then we have one four-level atom, e.g. the $|m\rangle = |3/2\rangle$ corresponds to the case when the highest (fourth) level is filled. If we are interested only in outcomes when the fourth or the second levels of the four-level atom are filled, then we can assume that we have a new set of two outcomes $\{\omega_1\}$ with probabilities $p_1 = p_{3/2} + p_{-1/2}$, $p_2 = p_{-3/2} + p_{1/2}$. If we are interested only in outcomes when the fourth and the third levels are filled, then we have another set of outcomes $\{\omega_2\}$ and their probabilities are $\tilde{p}_1 = p_{3/2} + p_{1/2}$, $\tilde{p}_2 = p_{-3/2} + p_{-1/2}$. The outcomes $\{\omega_1\}$ and $\{\omega_2\}$ are correlated. Hence, correlations in single qudit systems are between different combinations of outcomes.

With respect to our problem, the experiment could be designed so that we can measure the population only on certain levels, while others are not available to measure. This may be due to their short lifetime or a diversity in the frequency band reception. For example, we can measure only the first and the second levels of the four-level

atom and the third and the fourth levels are unobservable. Then we can think about the observable levels as about "artificial ancilla qubit" and about other two levels as about "artificial unobservable qubit". Thus, we can construct the observation model just like for the real two-qubit system.

V. FILTERING OF UNKNOWN SIGNALS

The problem of filtering of unknown signals from the mixture with a noise has a wide range of applications including control of linear and nonlinear systems. In the following, we consider that the Bloch vector of unobservable qubit θ_{S_2} is s_k , where k is the time step. The observable ancilla qubit will be characterized by parameter $c = \theta_{M_3}$. Hence, we can rewrite, for example, the second row in (4) as

$$s_k = \frac{s_{k-1} \pm c}{1 \pm cs_{k-1}}.$$

Let us rewrite the latter process under the assumption that c is small enough. Then the process will be the following

$$s_k = s_{k-1} \pm c(1 - s_{k-1}^2) + O(c)$$

or if we are interested in the system change only after N time steps we get

$$s_k = s_{k-1} + x_{k-1}c(1 - s_{k-1}^2), \quad (12)$$

where $x_k = x_{k+} - x_{k-}$, $N = x_{k+} + x_{k-}$ hold. We denote the plus and the minus outcomes as x_{k+} and x_{k-} , respectively.

Analogically to the previous example, for rotation matrix (6) and probabilities (7) we can write

$$\begin{aligned} s_k &= \frac{s_{k-1}(c+1) + c+1}{1 + cs_{k-1}} \\ &= \frac{(s_{k-1}(c+1) + c+1)(1 - cs_{k-1})}{(1 + cs_{k-1})^2} \\ &= s_{k-1} + (c+1)(1 - cs_{k-1}^2) + O(c). \end{aligned}$$

and the process is

$$s_k = s_{k-1} + (c+1)(1 - cs_{k-1}^2).$$

If we are interested in the system change only after N time steps, then the latter process can be rewritten as

$$s_k = s_{k-1} + x_{k-1}(c+1)(1 - cs_{k-1}^2). \quad (13)$$

We can get the whole class of processes like (12) and (13) depending on the choice of the evolution matrix. Thus, we have a partially observable Markov random sequence $(s_k, x_k)_{n \geq 1}$, where the sequence $s = (s_k)_{k \geq 1}$ is unobservable and the sequence $x = (x_k)_{k \geq 1}$ is observable. The connection between these variables is given by the following nonlinear or linear expression

$$x_k = \varphi(s_k, \eta_k), \quad (14)$$

where $(\eta_k \in \mathbb{R})_{k \geq 1}$ is an i.i.d random sequence, $(s_k)_{k \geq 1}$ is a Markov sequence and φ is some function. Realizations of random variables $s_k \in \mathcal{S}_k \subseteq \mathbb{R}$ and $x_k \in \mathcal{X}_k \subseteq \mathbb{R}$ are denoted by $s_1^k = (s_1, \dots, s_k)^T$ and $x_1^k = (x_1, \dots, x_k)^T$, respectively.

In case when (14) has the recursive linear form

$$\begin{aligned} s_k &= as_{k-1} + b\xi_k, \\ x_k &= As_k + B\eta_k, \end{aligned} \quad (15)$$

where $s_k, x_k \in \mathbb{R}$ for all k , ξ_k and η_k are mutually independent random variables with the standard Gaussian distribution,

$$\begin{aligned} s_0 &\in \mathcal{N}(0, \tilde{\sigma}^2), \quad \tilde{\sigma}^2 = \frac{b^2}{1-a^2} \\ s_n &\in \mathcal{N}(0, 1), \quad n = 1, 2, 3, \dots, \end{aligned}$$

coefficients A, B, a, b are given by real numbers and $|a| < 1$, the Kalman filter is applied as optimal method [39].

Remark V.1 In [42] the distribution of x_k is approximated by the Gaussian distribution $x_k \sim \mathcal{N}(Ncs_k; N)$ and the Kalman-like filter is applied. However, we would like to observe general case of nonlinear models that are more important for practice. In that case the Kalman filter does not provide the optimal filtration solution.

A. Nonlinear process

The optimal approach for nonlinear processes is proposed in [43]. To estimate s_n the optimal Bayesian estimator in form of the conditional mean

$$\hat{s}_n = \mathbb{E}(s_n | x_1^n) = \int_{\mathcal{S}_n} s_n w_n(s_n | x_1^n) ds_n, \quad (16)$$

has been used. The $w_n(s_n | x_1^n)$ is the posterior probability density function that satisfies the Stratonovich's recurrence equation [46] given by

$$\begin{aligned} w_1(s_1 | x_1) &= \frac{f(x_1 | s_1)p(s_1)}{\int_{\mathcal{S}_1} f(x_1 | s_1)p(s_1)ds_1}, \\ w_n(s_n | x_1^n) &= \frac{f(x_n | s_n)}{f(x_n | x_1^{n-1})} \\ &\cdot \int_{\mathcal{S}_{n-1}} p(s_n | s_{n-1})w_{n-1}(s_{n-1} | x_1^{n-1})ds_{n-1}, \quad n \geq 2. \end{aligned} \quad (17)$$

Here we denote as $p(s_n | s_{n-1})$ the transition probability density function of the Markov sequence $(S_n)_{n \geq 1}$ and $f(x_n | x_1^{n-1})$, $f(x_n | s_n)$ denote conditional densities.

Since the posterior density $w_n(s_n | x_1^n)$ depends on the unknown prior distribution function $p(s_1)$ and the transition probability $p(s_n | s_{n-1})$ of the Markov sequence $(s_n)_{n \geq 1}$, we cannot use formula (18) to estimate \hat{s}_n . To overcome this problem the general filtering equation is proposed in [44].

The exact coincidences of the general filtering equation for the unobservable Markov sequence (s_n) defined by a linear equation with a Gaussian noise) with Kalman filter and the conditional expectation $E(Q(s_n)|x_1^n)$ defined by Theorem of normal correlation [47] is proved in [45]. Thus, the general filtering equation is nothing else but the Kalman filter in case of linear model (15). However, for nonlinear processes the general filtering equation provides the optimal solution in contrast to the Kalman filter that cannot be applied to nonlinear models.

B. Equations of optimal filtering

Let us assume that the conditional density $f(x_n|s_n)$ belongs to the exponential family of distributions, i.e.

$$f(x_n|s_n) = \tilde{C}(s_n)h(x_n)\exp(T(x_n)Q(s_n)), \quad (18)$$

where $\tilde{C}(s_n)$ is a normalization constant and $h(x_n), T(x_n), Q(s_n)$ are known functions. The general filtration equation is

$$E(Q(s_n)|x_1^n) \cdot T'_{x_n}(x_n) = \left(\ln \left(\frac{f(x_n|x_1^{n-1})}{h(x_n)} \right) \right)'_{x_n} \quad (19)$$

Note that equation (19) does not contain explicit probabilistic characteristics $p(s_1)$ and $p(s_n|s_{n-1})$ of the unknown sequence (s_n) . This allows us to find the optimal estimator (16) knowing only observable quantities of x_1^n .

As an example of the exponential family (18) we can take the Gaussian density

$$f(x_n|s_n) = \frac{1}{\sqrt{2\pi}B} \exp \left(-\frac{(x_n - As_n)^2}{2B^2} \right). \quad (20)$$

Then the observation model is defined by the linear equation

$$x_n = As_n + B\eta_n, \quad (21)$$

where $\{\eta_n\}$ are independent identically distributed random variables with the Gaussian distribution and coefficients A and B are real numbers. Hence, we can rewrite equation (19) in a special form

$$E(s_n|x_1^n) = \frac{B^2}{A} \frac{f'_{x_n}(x_n|x_1^{n-1})}{f(x_n|x_1^{n-1})} + \frac{x_n}{A} \quad (22)$$

which is the Kalman filter.

Since the observation model given by equations (13) and (14) is nonlinear we propose to use general filtration equation (19) for the state estimation. It provides the optimal solution knowing only the observed random variables. We do not need any simplifications, linearizations or assumptions on the distribution of the observed random variables as it is done for example in [42].

VI. INEQUALITIES FOR QUANTUM TOMOGRAPHIC MUTUAL INFORMATION

The tomographic probability representation of spin (qudit) states is introduced in [48, 49]. In this representation qudit states with density matrices $\rho_S(k)$ and $\rho_M(k)$ are identified with spin-tomograms which are the probability distribution functions determined by the density operators of the states

$$\begin{aligned} \rho_S(k) &\Leftrightarrow \omega_S(m', U_S, k) = \langle m' | U_S \cdot \rho_S(k) \cdot U_S^\dagger | m' \rangle, \\ \rho_M(k) &\Leftrightarrow \omega_M(m, U_M, k) = \langle m | U_M \cdot \rho_M(k) \cdot U_M^\dagger | m \rangle, \end{aligned}$$

where $m', m = -j, -j+1, \dots, j$, $j = 0, 1/2, 1, \dots$ are spin projections and $U_S \equiv U(\theta_S, \varphi_S, \psi_S)$ and $U_M \equiv U(\theta_M, \varphi_M, \psi_M)$ are the rotation matrices of irreducible representations of $SU(2)$ - group

$$U(\theta_S, \varphi_S, \psi_S) = \begin{pmatrix} \cos \frac{\theta_S}{2} e^{i(\varphi_S + \psi_S)} & \sin \frac{\theta_S}{2} e^{i(\varphi_S - \psi_S)} \\ -\sin \frac{\theta_S}{2} e^{i(\psi_S - \varphi_S)} & \cos \frac{\theta_S}{2} e^{-i(\varphi_S + \psi_S)} \end{pmatrix}.$$

For density matrix ρ_{S+M} we can determine the tomogram

$$\begin{aligned} \rho_{S+M}(k+1) &\Leftrightarrow \omega_{S+M}(m', m, U_{S+M}, k+1) \\ &= \langle m' m | U_{S+M} \cdot \rho_{S+M}(k+1) \cdot U_{S+M}^\dagger | m' m \rangle. \end{aligned}$$

Rotation matrix U_{S+M} can be defined as the direct product of two matrices U_S and U_M

$$U_{S+M} = U(\theta_S, \varphi_S, \psi_S) \otimes U(\theta_M, \varphi_M, \psi_M).$$

The matrices U_S and U_M depend only on the Euler angles $\{\theta_i, \varphi_i, \psi_i\}$, $i \in \{S, M\}$ which determine directions of the quantization, e.g., points on the Bloch sphere. Hence, we use following notations $U_S = U(\vec{n}_S)$, $U_M = U(\vec{n}_M)$, where \vec{n}_S and \vec{n}_M determine directions of spin projection axes. Hence, the latter tomogram can be rewritten as

$$\begin{aligned} \omega_{S+M}(m', m, \vec{n}_S, \vec{n}_M, k+1) &= \langle m' m | U(\vec{n}_S) \otimes U(\vec{n}_M) \rho_{S+M}(k+1) U^\dagger(\vec{n}_S) \otimes U^\dagger(\vec{n}_M) | m' m \rangle \\ &= \langle m' m | U(\vec{n}_S) \otimes U(\vec{n}_M) W \rho_S(k) \otimes \rho_M(k) W^\dagger U^\dagger(\vec{n}_S) \otimes U^\dagger(\vec{n}_M) | m' m \rangle. \end{aligned} \quad (23)$$

It is the conditional probability of projections of spins m', m on vectors \vec{n}_S, \vec{n}_M on the Bloch sphere. Probability

function (23) has the property of a no-signaling. Hence, marginal probability distributions of the first system is given by

$$\omega_S(m', U_S, k+1) = \langle m' | U(\vec{n}_S) \rho_S(k+1) U^\dagger(\vec{n}_S) | m' \rangle = \langle m' | U(\vec{n}_S) \text{Tr}_M W \rho_S(k) \otimes \rho_M(k) W^\dagger U^\dagger(\vec{n}_S) | m' \rangle. \quad (24)$$

Since the ancilla qubit does not change in time we can write

$$\omega_M(m, U_M) = \langle m | U(\vec{n}_M) \rho_M U^\dagger(\vec{n}_M) | m \rangle.$$

Diagonal elements of the latter tomograms are $\omega_S(1, k+1)$, $\omega_S(2, k+1)$, $\omega_M(1)$, $\omega_M(2)$. By definition of the Shannon entropy, we can construct the tomographic entropies of the subsystems as

$$\begin{aligned} H_S(k+1) &= -\omega_S(1, k+1) \ln \omega_S(1, k+1) \\ &\quad - \omega_S(2, k+1) \ln \omega_S(2, k+1), \\ H_M(k+1) &= -\omega_M(1, k+1) \ln \omega_M(1, k+1) \\ &\quad - \omega_M(2, k+1) \ln \omega_M(2, k+1) \end{aligned}$$

Analogically, we can find the diagonal elements of $U_{S+M} \rho_{S+M} U_{S+M}^\dagger$ which are the tomograms $\omega_{S+M}(1, 1, k)$, $\omega_{S+M}(2, 2, k)$, $\omega_{S+M}(3, 3, k)$ and $\omega_{S+M}(4, 4, k)$. The Shannon entropy of the combined system is

$$\begin{aligned} H_{S+M}(k) &= -\omega_{S+M}(1, 1, k) \ln \omega_{S+M}(1, 1, k) \\ &\quad - \omega_{S+M}(2, 2, k) \ln \omega_{S+M}(2, 2, k) \\ &\quad - \omega_{S+M}(3, 3, k) \ln \omega_{S+M}(3, 3, k) \\ &\quad - \omega_{S+M}(4, 4, k) \ln \omega_{S+M}(4, 4, k). \end{aligned}$$

The Shannon information depending on the time step is

$$I(k) = H_S(k) + H_M(k) - H_{S+M}(k) \geq 0.$$

In case when the state of the ancilla qubit does not change in time the latter inequality can be rewritten as

$$I(k) = H_S(k) + H_M - H_{S+M}(k) \geq 0. \quad (25)$$

A. Example

As an example let us take the Werner state [50]. The two-qubit system or the single qudit system with the spin $j = 3/2$ can be described by the following Werner density matrix

$$\rho_{S+M}(0) = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}, \quad (26)$$

where the parameter p satisfies the inequality $-1/3 \leq p \leq 1$. The parameter domain $1/3 < p \leq 1$ corresponds to the entangled state. The reduced density matrices of the first and the second qubits (or the "artificial qubits") are

$$\rho_S(0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \rho_M = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Let us take evolution matrix (5) with $\varphi = \pi/8$ and the angles of the rotation matrices $\{\theta, \varphi, \psi\} = \{2\pi, \pi, \pi\}$ both for U_S and U_M . On the time step $k = 0$ and $k = 1$ the information (25) is

$$\begin{aligned} I(0) &= 2 \ln 2 - \ln \left(\frac{1-p}{4} \right) \frac{(p-1)}{2} + \ln \left(\frac{1+p}{4} \right) \frac{(p+1)}{2}, \\ I(1) &= \ln 2 + \ln \left(\frac{(p + \sqrt{2}p + 1)}{4} \right) \frac{(p + \sqrt{2}p + 1)}{4} + \ln \left(\frac{(p - \sqrt{2}p + 1)}{4} \right) \frac{(p - \sqrt{2}p + 1)}{4} \\ &\quad - \ln \left(\frac{1-p}{4} \right) \frac{(p-1)}{2} + \ln \left(\frac{1 - \frac{\sqrt{2}p}{2}}{2} \right) \frac{(\sqrt{2}p - 2)}{4} - \left(\frac{1 + \frac{\sqrt{2}p}{2}}{2} \right) \frac{(\sqrt{2}p + 2)}{4}. \end{aligned}$$

The information (25) against parameter p is shown in Fig. 1 for time steps $k = \{0, 1, 2\}$.

VII. CONCLUSION

To conclude let us point out the main results of our work. Using the invertible method of indices we extended the observation model based on the indirect measurement known for the two-qubit system to the single qudit

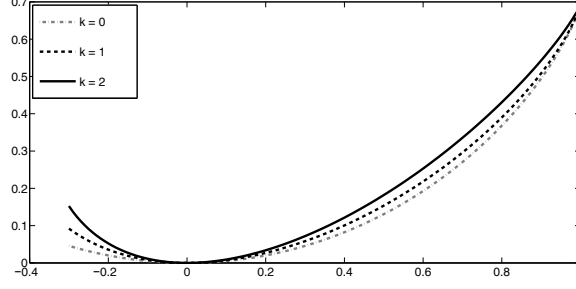


FIG. 1. Information (25) against parameter p depending on the time step $k = \{0, 1, 2\}$

system. Since the models can be nonlinear the known Kalman filter approach does not give the optimal solution. Hence, we propose to use for these nonlinear quantum models the general filtering equation that gives us the optimal solution for the nonlinear filtering problems. It provides the optimal solution to the state estimation problem knowing only the observed random variables. In contrast to the known in the literature state estimation methods we do not need any simplifications, linearizations or assumptions on the distribution of the observed random variables. Therefore, having the nonlinear observation model one can forget about its physical nature and apply the filtering method described above.

Moreover, the tomographic approach is used to write the Shannon entropies for the examined states depending on the time step of the quantum observation model. Using the latter entropies, the time step dependent information inequality is obtained. As an example we select the Werner state to show the time evolution of the information.

ACKNOWLEDGEMENTS

The study in Sections 2 and 3 by Markovich L.A. was supported by the Russian Science Foundation grant (14-50-00150).

VIII. APPENDIX

The numerator elements (3) for evolution matrix (5) are

$$\begin{aligned} \widetilde{\rho}_S(1, 1) = & \frac{1}{8} \left(-(\theta_{m_3} - 1)(\theta_{s_3} + 1) + \cos \varphi^* \right. \\ & \cdot (\cos \varphi(\theta_{m_3} + 1)(\theta_{s_3} + 1) + \sin \varphi(\theta_{m_1} + \theta_{m_2}i)(\theta_{s_1} + \theta_{s_2}i)) \\ & + \sin \varphi^*(\sin \varphi(\theta_{m_3} - 1)(\theta_{s_3} - 1) + \cos \varphi(\theta_{m_1} - \theta_{m_2}i) \\ & \cdot (\theta_{s_1} - \theta_{s_2}i)) + \cos \varphi^*(\theta_{s_3} + 1)(\theta_{m_1} + \theta_{m_2}i) \\ & - \sin \varphi^*(\theta_{m_3} - 1)(\theta_{s_1} - \theta_{s_2}i) + \cos \varphi(\theta_{s_3} + 1) \\ & \cdot (\theta_{m_1} - \theta_{m_2}i) - \sin \varphi(\theta_{m_3} - 1)(\theta_{s_1} + \theta_{s_2}i) \Big), \end{aligned}$$

$$\begin{aligned} \widetilde{\rho}_S(1, 2) = & \frac{1}{8} \left(\cos \varphi^*(\sin \varphi(\theta_{m_3} - 1)(\theta_{s_3} - 1) \right. \\ & + \cos \varphi(\theta_{m_1} - \theta_{m_2}i)(\theta_{s_1} - \theta_{s_2}i)) - \sin \varphi^* \\ & \cdot (\cos \varphi(\theta_{m_3} + 1)(\theta_{s_3} + 1) + \sin \varphi(\theta_{m_1} + \theta_{m_2}i) \\ & \cdot (\theta_{s_1} + \theta_{s_2}i)) - (\theta_{m_1} + \theta_{m_2}i)(\theta_{s_1} - \theta_{s_2}i) - \cos \varphi^* \\ & \cdot (\theta_{m_3} - 1)(\theta_{s_1} - \theta_{s_2}i) - \sin \varphi^*(\theta_{s_3} + 1)(\theta_{m_1} + \theta_{m_2}i) \\ & + \cos \varphi(\theta_{m_3} + 1)(\theta_{s_1} - \theta_{s_2}i) \\ & \left. - \sin \varphi(\theta_{s_3} - 1)(\theta_{m_1} + \theta_{m_2}i) \right), \end{aligned}$$

$$\begin{aligned} \widetilde{\rho}_S(2, 1) = & \frac{1}{8} \left(-\cos \varphi^*(\sin \varphi(\theta_{m_3} + 1)(\theta_{s_3} + 1) \right. \\ & - \cos \varphi(\theta_{m_1} + \theta_{m_2}i)(\theta_{s_1} + \theta_{s_2}i)) + \sin \varphi^* \\ & \cdot (\cos \varphi(\theta_{m_3} - 1)(\theta_{s_3} - 1) - \sin \varphi(\theta_{m_1} - \theta_{m_2}i) \\ & \cdot (\theta_{s_1} - \theta_{s_2}i)) + (\theta_{m_1} - \theta_{m_2}i)(\theta_{s_1} + \theta_{s_2}i) \\ & + \cos \varphi^*(\theta_{m_3} + 1)(\theta_{s_1} + \theta_{s_2}i) - \sin \varphi^*(\theta_{s_3} - 1) \\ & \cdot (\theta_{m_1} - \theta_{m_2}i) - \cos \varphi(\theta_{m_3} - 1)(\theta_{s_1} + \theta_{s_2}i) \\ & \left. - \sin \varphi(\theta_{s_3} + 1)(\theta_{m_1} - \theta_{m_2}i) \right), \end{aligned}$$

$$\begin{aligned} \widetilde{\rho}_S(2, 2) = & \frac{1}{8} \left(-(\theta_{m_3} + 1)(\theta_{s_3} - 1) + \cos \varphi^* \right. \\ & \cdot (\cos \varphi(\theta_{m_3} - 1)(\theta_{s_3} - 1) - \sin \varphi(\theta_{m_1} - \theta_{m_2}i) \\ & \cdot (\theta_{s_1} - \theta_{s_2}i)) + \sin \varphi^*(\sin \varphi(\theta_{m_3} + 1)(\theta_{s_3} + 1) \\ & - \cos \varphi(\theta_{m_1} + \theta_{m_2}i)(\theta_{s_1} + \theta_{s_2}i)) - \cos \varphi^*(\theta_{s_3} - 1) \\ & \cdot (\theta_{m_1} - \theta_{m_2}i) - \sin \varphi^*(\theta_{m_3} + 1)(\theta_{s_1} + \theta_{s_2}i) \\ & - \cos \varphi(\theta_{s_3} - 1)(\theta_{m_1} + \theta_{m_2}i) \\ & \left. - \sin \varphi(\theta_{m_3} + 1)(\theta_{s_1} - \theta_{s_2}i) \right) \end{aligned}$$

and the denominator is the following

$$\begin{aligned} & Tr(Tr_M(\rho_{S+M}(k+1)A_x)) = \\ & = \frac{1}{4} \left(\cos(\varphi^* - \varphi) + \theta_{m_1}(\cos \varphi + \cos \varphi^*) \right. \\ & + \theta_{s_2}(\sin \varphi - \sin \varphi^*)i + \theta_{m_2}\theta_{s_3}(\cos \varphi^* - \cos \varphi)i \\ & \cdot \theta_{m_3}\theta_{s_3}(\cos(\varphi^* - \varphi) - 1) - \theta_{m_3}\theta_{s_1}(\sin \varphi^* + \sin \varphi) \\ & \left. - (\theta_{m_1}\theta_{s_2} + \theta_{m_2}\theta_{s_1})\sin(\varphi^* - \varphi)i + 1 \right). \end{aligned}$$

The numerator elements (3) for evolution matrix (6) are

$$\begin{aligned}
\widetilde{\rho}_S(1,1) &= \frac{1}{8} \left(\cos \varphi^* (\cos \varphi (\theta_{m_3} + 1) (\theta_{s_3} + 1) + \sin \varphi \right. \\
&\quad \cdot (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) + \sin \varphi^* (\sin \varphi (\theta_{m_3} - 1) \\
&\quad \cdot (\theta_{s_3} - 1) + \cos \varphi (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(1,2) &= \frac{1}{8} \left(\cos \varphi^* (\sin \varphi (\theta_{m_3} - 1) (\theta_{s_3} - 1) + \cos \varphi \right. \\
&\quad \cdot (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) - \sin \varphi^* (\cos \varphi (\theta_{m_3} + 1) \\
&\quad \cdot (\theta_{s_3} + 1) + \sin \varphi (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(2,1) &= \frac{1}{8} \left(-\cos \varphi^* (\sin \varphi (\theta_{m_3} + 1) (\theta_{s_3} + 1) - \cos \varphi \right. \\
&\quad \cdot (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) + \sin \varphi^* (\cos \varphi (\theta_{m_3} - 1) \\
&\quad \cdot (\theta_{s_3} - 1) - \sin \varphi (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(2,2) &= \frac{1}{8} \left(\cos \varphi^* (\cos \varphi (\theta_{m_3} - 1) (\theta_{s_3} - 1) - \sin \varphi \right. \\
&\quad \cdot (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) + \sin \varphi^* (\sin \varphi (\theta_{m_3} + 1) \\
&\quad \cdot (\theta_{s_3} + 1) - \cos \varphi (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) \Big),
\end{aligned}$$

and the denominator is

$$\begin{aligned}
&\frac{1}{4} \left(\cos(\varphi^* - \varphi) (1 + \theta_{m_3} \theta_{s_3}) \right. \\
&\quad \left. - (\theta_{m_1} \theta_{s_2} + \theta_{m_2} \theta_{s_1}) \sin(\varphi^* - \varphi) i \right).
\end{aligned}$$

For the real φ the latter formulas may be reduced to

$$\begin{aligned}
\widetilde{\rho}_S(1,1) &= \frac{1}{8} \left(\cos \varphi (\cos \varphi (\theta_{m_3} + 1) (\theta_{s_3} + 1) + \sin \varphi \right. \\
&\quad \cdot (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) + \sin \varphi (\sin \varphi (\theta_{m_3} - 1) \\
&\quad \cdot (\theta_{s_3} - 1) + \cos \varphi (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(1,2) &= \frac{1}{8} \left(\cos \varphi \sin \varphi (\theta_{m_3} - 1) (\theta_{s_3} - 1) + \cos \varphi \right. \\
&\quad \cdot (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) \sin \varphi (\cos \varphi (\theta_{m_3} + 1) \\
&\quad \cdot (\theta_{s_3} + 1) + \sin \varphi (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(2,1) &= \frac{1}{8} \left(-\cos \varphi (\sin \varphi (\theta_{m_3} + 1) (\theta_{s_3} + 1) - \cos \varphi \right. \\
&\quad \cdot (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) + \sin \varphi (\cos \varphi (\theta_{m_3} - 1) \\
&\quad \cdot (\theta_{s_3} - 1) - \sin \varphi (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) \Big), \\
\widetilde{\rho}_S(2,2) &= \frac{1}{8} \left(\cos \varphi (\cos \varphi (\theta_{m_3} - 1) (\theta_{s_3} - 1) - \sin \varphi \right. \\
&\quad \cdot (\theta_{m_1} - \theta_{m_2} i) (\theta_{s_1} - \theta_{s_2} i)) + \sin \varphi (\sin \varphi (\theta_{m_3} + 1) \\
&\quad \cdot (\theta_{s_3} + 1) - \cos \varphi (\theta_{m_1} + \theta_{m_2} i) (\theta_{s_1} + \theta_{s_2} i)) \Big),
\end{aligned}$$

-
- [1] M. Armen, J. Au, J. Stockton, A. Doherty, and H. Mabuchi, Phys. Rev. Lett. **89**, 133602 (2002).
 - [2] A. Barchielli, Quantum Opt. **2**, 423 (1990).
 - [3] L. Bouten, R. van Handel, and M. R. James, SIAM J. Control Optim **46**, 2199 (2007).
 - [4] V. Belavkin, Automat. and Remote Control **44**, 178 (1983).
 - [5] A. Branczyk, P. Mendonca, A. Gilchrist, A. Doherty, and S. Bartlett, Phys. Rev. A **75**, 012329 (2007).
 - [6] A. Soare, H. Ball, D. Hayes, X. Zhen, M. C. Jarratt, J. Sastrawan, H. Uys, and M. J. Biercuk, Phys. Rev. A **89**, 042329 (2014).
 - [7] Z. Yan, X. Jia, C. Xie, and K. Peng, Phys. Rev. A **84**, 062304 (2011).
 - [8] L. Bouten, R. van Handel, and M. R. James, SIAM Rev. **51**, 239 (2009).
 - [9] G. Smith, A. Silberfarb, I. Deutsch, and P. Jessen, Phys. Rev. Lett. **97**(18), 180403 (2006).
 - [10] R. Cook, C. A. Riofrio, and I. Deutsch, Phys. Rev. A **90**, 032113 (2014).
 - [11] J. Ralph, K. Jacobs, and C. Hill, Phys. Rev. A **84**, 052119 (2011).
 - [12] S. Wang and T. Byrnes, Phys. Rev. A **94**, 033620 (2016).
 - [13] C. Ahn, A. Doherty, and A. Landahl, Phys. Rev. A **65**, 042301 (2002).
 - [14] M. Sarovar, C. Ahn, and K. J. G. J. Milburn, Phys. Rev. A **69**, 052324 (2004).
 - [15] C. D'Helon, A. Doherty, M. James, and S. Wilson, 45th IEEE Conference on Decision and Control CDC 2006 United States: (IEEE) Institute of Electrical and Electronics Engineers. (2006).
 - [16] C. D'Helon and M. James, Phys. Rev. A **73**, 053803 (2006).
 - [17] J. Gough and J. Kostler, Communications on Stochastic Analysis **4**(4), 505 (2010).
 - [18] C. Aguirre and R. Mendes, IET Signal Processing **8**(1), 67 (2014).
 - [19] R. Mendes, H. Mendes, and T. Araujo, Phys. A **450**, 1 (2016).
 - [20] M. Man'ko, V. Man'ko, and R. Mendes, J. of Phys. A **34**, 8321 (2001).
 - [21] L. Ruppert and K. Hangos, Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems, 2049 (2010).
 - [22] A. Korotkov and A. Jordan, Phys. Rev. Lett. **97**, 166805 (2006).
 - [23] Y. Kim, Y. Cho, Y. Ra, and Y. Kim, Optics Express **17**(14), 11978 (2009).
 - [24] A. Jordan, B. Trauzettel, and G. Burkard, Phys. Rev. B **76**, 155324 (2007).

- [25] J. Reháček, B. Englert, and D. Kaszlikowski, Phys. Rev. A **70**(11), 052321 (2004).
- [26] K. Jacobs and D. Steck, Contemporary Physics **47**(5), 279 (2007).
- [27] E. Schrodinger, Naturwissenschaften **23**, 807 (1935).
- [28] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. **23** (15), 880 (1969).
- [29] V. Man'ko and A. Yurkevich, J. Russ. Laser Res. **34** (5), 463 (2013).
- [30] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
- [31] V. Chernega and V. Man'ko, J. Russ. Laser Res. **29**, 505 (2008).
- [32] V. Chernega and O. Man'ko, J. Russ. Laser Res. **35** (1), 27 (2014).
- [33] M. Man'ko and V. Man'ko, Physica Scripta **T160** (2014).
- [34] M. Manko and V. Manko, (unpublished) arXiv:1409.4221 (2014).
- [35] V. Man'ko and L. Markovich, J. Russ. Laser Res. **35**(5), 518 (2014).
- [36] V. I. Man'ko and L. Markovich, J. Russ. Laser Res. **36**(4), 343 (2015).
- [37] M. Zukowski, A. Dutta, and Z. Yin, (unpublished) arXiv:1411.5986 (2014).
- [38] V. I. Man'ko and L. Markovich, Phys. A **4**, 266 (2016).
- [39] R. Kalman, Journal of Basic Engineering **82**(1), 35 (1960).
- [40] F. Yang, Z. Wang, and Y. Hung, IEEE Transactions On Automatic Control **47**, 1179 (2002).
- [41] O. Cappe, E. Moulines, and T. Ryden, *Inference in Hidden Markov Models* (Springer, 2005).
- [42] L. Ruppert, Technical report SCL **001** (2012).
- [43] R. Stratonovich, Theory of Probability and its Applications **5**, 156 (1960).
- [44] A. Dobrovidov, Automat. and Remote Control **44**(6), 757 (1983).
- [45] L. A. Markovich, Lith. Math. J. **55** (3), 413 (2015).
- [46] R. Stratonovich, *Conditional Markovian processes and their application to the optimal control theory* (Moscow Univ. Press, (in Russian)., 1966).
- [47] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes: II. Applications* (Springer, 2001).
- [48] V. V. Dodonov and V. I. Man'ko, Phys. Lett. A **229**, 335 (1997).
- [49] V. I. Man'ko and O. V. Man'ko, JETP **85** (3), 430 (1997).
- [50] R. Werner, Phys. Rev. A **40**, 4277 (1989).

(Dated: January 24, 2017)

Abstract

